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FUNDAMENTAL IDEAS IN THE HISTORY OF MATHEMATICS

BY

G. A. MILLER, *University of Illinois.*

In Webster's *New International Dictionary*, second edition, 1934, there appears under the entry "algebra" the following striking assertion, "The essential difference between arithmetic and algebra is that the former deals with concrete quantities, while the latter deals with symbols whose values may be any out of a given field." This statement does not appear in the earlier editions of this widely used dictionary and hence it might at first appear to represent a modern view in regard to the difference between the extensive domains of mathematics which are now commonly classed throughout the civilized world under the somewhat vague terms of arithmetic and algebra. On the contrary, it represents a great lack of insight into the fundamental facts relating to the development of mathematics during the time extending back to the earliest extant mathematical treatises, such as the now well-known Egyptian work commonly called "Rhind Mathematical Papyrus," which is by far the most comprehensive ancient Egyptian mathematical work now known.

This particular work is supposed to have been written about 1700 B. C. but is explicitly based on an earlier work and begins with a table relating entirely to abstract numbers, viz., to the expressing of the common fractions in the form of $2/n$, where n represents the successive odd integers from 5 to 101, in terms of the reciprocals of different natural numbers, and relates to the following elementary theorem: *An arbitrary rational number can be expressed as the sum of the reciprocals of a finite set of distinct integers having the same sign and such that the smallest absolute value of the integers of the set is arbitrarily large and that the number of these integers exceeds any assigned natural number.** This theorem constitutes a theoretic basis of much of the mathematical work during ancient and medieval times even if it does not seem to have been explicitly formulated before 1931.

It furnishes further evidence of the fact that even many of the earliest known developments in arithmetic relate to abstract quantities. The fact th[at] in all the civilized languages of the world there appears a term for one which is unrelated to the term for the number two, shows that the i[...]

* G. A. Miller, *Ame[rican] Mathematical Monthly*, vol. 38 (1931), p. 194.

fraction is not only prehistoric but that it arose as an individual notion and not as a notion which was at first related to the natural numbers. So that there are *natural fractions* just as definitely as there are natural integers. The common fractions which arose in connection with the natural numbers have recently been called *algorithmic fractions* by O. Neugebauer in his *Vorlesungen uber Geschichte der Antiken Mathematischen Wissenschaften*, volume 1 (1934), page 88. At this place there appears also an interesting account of the complementary fractions, that is, fractions of the form $1-1/n$, which include 2/3 and play such a prominent rôle in ancient mathematics, and relate entirely to abstract arithmetical notions.

The main object of the preceding remarks is to exhibit instances tending to show that one of the profoundest facts of mathematical history is that *mathematics has not become relatively more abstract since the most ancient extant mathematical treatises were written.* There has been a growing tendency in mathematics towards greater generalization but not towards greater abstraction since the appearance of these treatises. Algebra differs from arithmetic *because it deals* with relatively more general ideas than arithmetic, and there is not an abrupt passage from the one to the other but a gradual one. It is obviously true that abstract ideas are more general than concrete ones but the passage from concrete numbers to abstract numbers represents only one stage in the generalization of the number concept. Other stages in their generalization are represented by the recognition that such a fraction as 7/5 belongs really to the domain of distinct numbers and is not merely an indicated division, or a binary number, and that negative numbers and complex numbers really form a part of our number system.

According to page 150 of the volume by O. Neugebauer cited above, the step of regarding such a fraction as 7/5 as an individual number, viz., the individual rational number in our common domain of rational numbers now commonly represented by this symbol, was not taken anywhere in the extant mathematics of antiquity. This step was preceded by the geometric notion of regarding numbers as represented by points on a line and the group theory notion of dealing with domains composed of the elements resulting from certain operations on the given elements of the domains. Notwithstanding the great simplicity of these ideas as they appear in our modern mathematical text-books one fails completely in understanding the fundamental ideas of the history of our subject unless one considers the evidences of the slow and halting movements in the progress of mathematical developments towards ideas which are now supposed to be mastered at an early stage of the mathematical attainments of our students. As a matter of fact they are ideas which are commonly absorbed at first by the student but not comprehended by him for a long time.

The results of the great co-operation between abstract and concrete mathe-
[illegible]tics are now commonly denoted by the terms pure and applied mathematics
[illegible]ctively. In the large German mathematical encyclopaedia which began to
[illegible] in 1898 and was recently announced as completed, but which does not
[illegible]e the originally planned volume 7 on the [illegible]phy, teaching, and

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history of mathematics, more than half of the space is devoted to applied mathematics. While it is reasonable to assume that the earliest developments in mathematics related to concrete quantities it is now known that in very early times the Babylonians developed a type of pure mathematics which seems to have no direct connection with applications and is regarded as algebraic by O. Neugebauer in the volume to which we referred above, pages 2 and 205. On the contrary, E. Bortolotti expressed the view that it should not be regarded as algebraic since the number system employed therein is confined to a very small part of the domain of rational numbers and hence was very insufficient to solve even the quadratic equation except in very special cases. Cf. *Bolletino della Unione Matematica Italiana*, volume 14 (1935), page 112.

It should be observed in this connection that in the history of mathematics it is commonly assumed that the early developments in algebra are much older than the development of a number system which is adequate for the solution of the general quadratic equation. In fact, they are much older than the earliest satisfactory theory of the operations with negative numbers. One of the main reasons for saying that the ancient Babylonians developed a mathematics which may be regarded as partly algebraic is that they solved quadratic equations by general methods which are still used at times. Their solutions of cubic and biquadratic equations, on the contrary, are not in accord with our modern general formulas for this purpose. In the case of cubic equations they used the method of reducing them to the form $y^3+y^2=k$ and then employed tables of the form n^2+n^3 for their solution. A weak element in their solutions of equations is that the equations solved by them seem to have been constructed, at least in many cases, to fit known values of the variables involved therein. This is a fundamental fact in the history of algebra and weakens very much the esteem which we would otherwise have of the algebraic work of the Babylonians.

Here as in many other places in the history of mathematics it is impossible to make much progress without clear definition of the terms involved. People seem to have very different ideas as regards what constitutes an algebraic development and hence it is only natural that there is no general agreement with respect to the question whether some of the methods used by the Babylonians about 2000 B. C. are algebraic or whether they are all confined to arithmetic. It is, however, well established that they are now regarded as partly algebraic by various careful writers on the ancient history of our subject, and hence the history of algebra has been greatly extended backwards during the last decade. In view of these extensions most of our general histories of mathematics are in need of revision, especially with respect to the parts dealing with ancient mathematics while a decade ago little progress was anticipated on the part of many students of our subject as regards this history.

The extent of the material relating to ancient Egyptian mathematics is [illegible] large but several hundred thousand clay tablets relating to the ar[illegible] Babylonian civilization are now stored in the various museums of th[illegible]

It is expected that a relatively small number of these are devoted to mathematics and hence the history of ancient Babylonian mathematics differs widely from that of modern mathematics, with respect to the number of works that deserve examination therein. In the history of modern mathematics there seems to be an endless amount of material since new developments are continually published. These frequently throw new light on some of the older developments. It is therefore clear that a complete history of mathematics is impossible as long as mathematical progress is being made, and there seems to be no evidence tending to show that this progress will ever cease.

A fundamental idea in the history of mathematics is that the reader should be encouraged to draw his own conclusions from the evidences presented and should not be confronted merely with conclusions which he is expected to accept. For instance, it is of very little value to be told that R. Descartes invented analytic geometry and that the subject of calculus was founded by I. Newton and G. W. Leibniz. Many students who are familiar with these conclusions would be unable to furnish substantial evidences in their support. Similarly, various students who would be ready to assert that J. Napier invented logarithms would be unable to defend their assertion against the counter claims that the theory of logarithms was known before the time of J. Napier and that the first table of logarithms was constructed by H. Briggs. Considerations of evidences imply a much wider acquaintance with the facts involved than the mere acceptance of conclusions but they also constitute a firmer basis for further advances.

A fundamental idea in the history of mathematics is that there always has been a tendency towards deviating from the ordinary language and the ordinary grammatical constructions in speaking about mathematical questions. On page 70 of the volume by O. Neugebauer to which we referred above it is noted that in ancient Babylonian mathematical literature one can already see a tendency towards a special mathematical language. In modern times we may cite an instance of this in the use of the term every two or every pair. The early writers commonly said that a set of distinct substitutions constitutes a group if it contains the product of every two of them and the square of every one, but on page 22 of C. Jordan's noted *Traitè des substitutions*, 1870, the latter part of this statement is omitted, and hence it is implicitly assumed here that the expression "the product of every two" implies that in this product the factors may be equal to each other. His example was unfortunately followed by many later writers.

It is sometimes said that the number of complete pairs of n distinct elements is $n(n+1)/2$ while the number of simple pairs is $n(n-1)/2$. Cf. *Enciclopedia delle Mathematiche Elementari,* volume 1, part 2 (1932), page 9. The vagueness re- ulting from the different uses of the term "every two" in group theory was sidered by the present writer in an article entitled "Sets of distinct operators ving all the products but not all the squares." *Proceedings of the National* v *of Sciences*, volume 18 (1932), pages 100-2. In particular, it is proved

here that if a set of distinct operators has the property that it involves the product of every simple pair of them irrespective of the order of the two factors, and generates a non-abelian group but does not involve all the operators of his group, then it is composed of the identity and the six operators of order 4 of the quaternion group. An instance of the deviation from ordinary grammatical usage in more elementary modern mathematics is seen in the omission of the abbreviation period in such terms as log and sin.

Various terms used in mathematics have acquired technical meanings which differ from the meanings commonly assigned to them in ordinary language. For instance, each of the terms *function* and *group* which are widely employed in the modern mathematical literature, has acquired a meaning therein which is much more definite than that assigned to it in ordinary language. Unfortunately the latter of these terms is still frequently employed in mathematical writings with somewhat vague meanings, as may be seen from its mathematical definitions in the dictionary to which we referred at the opening of the present article. See also the entry "distribution" in their dictionary. Some terms, such as infinitesimal, have been employed with different meanings in the mathematical literature at different periods. This constitutes an element of considerable difficulty in the history of mathematics in view of the fact that writers on this subject have frequently failed to exhibit clearly the meanings with which they used certain terms. Technical terms should always be employed in the modern history of our subject with their commonly accepted modern meanings unless the contrary is explicitly stated. This may be regarded as a postulate as regards modern historical writings.

A fundamental idea in the history of mathematics is that conflicting statements abound therein and hence the reader must adopt the policy of not accepting statements without verification. For instance, in volume 1 (1923), of D. E. Smith's *History of Mathematics* it is stated under the name Heron (page 126) that "he was able to solve the equation which we write in the form $ax^2+bx=c$, so that the general quadratic as we know it to-day was thus fully mastered by the Greek mathematicians." The reader who bears in mind that the ancient Greeks did not even use negative numbers and were entirely unacquainted with our ordinary system of complex numbers, and hence could not have solved the general quadratic equation even geometrically, will naturally wonder what meaning the author of this quotation could have had in mind, in case the reader is not ready to assume that the author was densely ignorant of the subject on which he was writing. A thoughtful reading of other parts of this volume and of volume 2 of the same work will naturally incline him to adopt this assumption as the most plausible. Cf. *Journal of the Indian Mathematical Society*, volume 18 (1930), pages 265-270.

Writers on the history of mathematics have usually dealt very largely with hypotheses. It is important for the beginner to differentiate between working hypotheses and established facts. In the preface to the work by O. Neugebauer to which we referred above it is stated (page VIII) that it seemed to the author

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more important to induce others to consider the questions involved than to conceal an error in his own views by cautious silence. Unfortunately, the hypotheses of one writer are frequently stated as facts by later authors. For instance, Herodotus stated as a hypothesis that geometry originated in Egypt as a result of the fact that the annual overflow of the Nile made it necessary to remeasure portions of land. Later writers often stated this as a fact with respect to the origin of geometry. It is questionable whether a readable history of mathematics could be written without the use of hypotheses, but readers should be careful to distinguish between hypotheses and established facts.

Hypotheses constitute a convenient method for grouping facts which might otherwise appear as unrelated and hence they simplify the study of the history of our subject. When these facts accumulate and no facts are presented which seem to contradict the hypothesis under consideration it is natural that this particular hypothesis is later commonly accepted as a fact. As an instance of such a hypothesis we may cite the rotation of the earth on its axis. On the other hand, the wide difference between mathematical induction and ordinary induction should always be observed in the study of the history of mathematics and it should be noted that this history may be made a very useful aid for the study of the subject of mathematics itself.

ON A PROPERTY OF THE FOCAL SURFACE

BY

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§ **1.** Consider the congruence of curves $F_r(x, y, z, a, b) = 0$, $r = (1, 2)$, each curve being regarded as the complete intersection of two surfaces. We may suppose that F_1 and F_2 are free from irrational and transcendental expressions (we shall allow such expressions in the u, v forms). For the definitions and general properties of the focal surface, the reader is referred to well-known works on Differential Geometry: The present note is in connection with the following property: "Any surface of the congruence touches the focal surface at the foci of any of its curves."*

The exceptions to this result do not appear to have been properly recognised and discussed. When F_1 and F_2 both involve a and b in *implicit* form, there exists in the general case a ∞^1 surfaces of the congruence which do not touch the focal surface.

* Forsyth: *Diff. Geom.* § 270. Weatherburn: *Diff. Geom.* I. § 104.

The directions of the normal to the focal surface at a focal point are given by

$$\left(k\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x},\ k\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y},\ k\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \right),$$

where
$$k = \frac{\partial F_1}{\partial a} \Big/ \frac{\partial F_2}{\partial a} = \frac{\partial F_1}{\partial b} \Big/ \frac{\partial F_2}{\partial b}.$$

The directions of the normal at any point on the surface of the congruence generated by the curves $F_r\Big(x,\ y,\ z,\ a,\ \phi'(a)\Big) = 0,\ r = 1,\ 2$ are given by

$$\left(\mu\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x},\ \text{etc., etc.} \right) \text{ where } \mu = \left(\frac{\partial F_1}{\partial a} + \frac{\partial F_1}{\partial b}\phi'(a) \right) \Big/ \left(\frac{\partial F_2}{\partial a} + \frac{\partial F_2}{\partial b}\phi'(a) \right)$$

The general property under discussion follows from the fact that at a focal point $\mu = k$.

It can however happen that μ is indeterminate i.e., that at certain points on the surface of the congruence, we have

$$\frac{\partial F_r}{\partial a} + \frac{\partial F_r}{\partial b}\ \phi'(a) = 0, \qquad (r = 1,\ 2).$$

If we eliminate $x,\ y,\ z$ between the four equations

$$\left.\begin{array}{l} F_r(x,\ y,\ z,\ a,\ b) = 0 \\ \dfrac{\partial F_r}{\partial a} + \dfrac{\partial F_r}{\partial b}\dfrac{db}{da} = 0 \end{array}\right\},\ (r = 1,\ 2) \qquad \text{(A)}$$

we obtain a differential equation $\lambda\left(a,\ b,\ \dfrac{db}{da}\right) = 0$. If any solution $b = \phi(a)$ of this differential equation be used to determine a surface of the congruence, this surface will not necessarily touch the focal surface.

[The arguments may be put in geometrical language. Let P be a focal point on a curve of the congruence lying on some one of the surfaces of the congruence. It is known that the curve touches the focal surface at P. The surface of the congruence meets the focal surface along a locus of focal points. The tangent to this curve-locus at P, and the tangent to the curve of the congruence at P are therefore two different tangent lines either to the focal surface, or to the surface of the congruence considered. Contact at P between the two surfaces is thus established in the general case. The difficulty comes in when the curve of the congruence touches the locus of the focal points on the surface, for in this case, the two surfaces have

only one tangent line in common. Writing down the conditions that a system of curves selected out of the congruence may have an enveloping curve, we arrive at the equations (A).]

The surface generated by a singly infinite system of curves

$$F_r\left(x, y, z, a, \phi(a)\right) = 0, \qquad r = 1, 2$$

can be guaranteed to touch the focal surface at the focal points of these curves only when these points do not satisfy (A)—which means, in general, that the locus of the focal points is not an envelope of the curves.

§ **2.** All that has been stated above is that the usual proof breaks down for the ∞^1 surfaces of the congruence derived from (A). Further examination is desirable to ascertain definitely whether these surfaces do or do not touch the focal surface.

In the case of rectilinear congruences, the surfaces derived from (A) are the developable surfaces of the congruence, and the locus of the focal points on any one of these surfaces is its edge of regression and is hence a unodal line. The unodal tangent plane at any focal point may not necessarily touch the focal surface thereat. This explains the exceptional case under consideration, for rectilinear congruences. I have given, however, two examples at the end wherein the developable surfaces are planes, so that the above explanation does not hold, and yet the planes do not touch the focal surface.

A similar explanation holds for congruences of circles, since the locus of the focal points on any surface of the congruence derived from (A) is known to be a double line.*

The general congruence also behaves likewise, but exceptions are not impossible. If (x, y, z) and $(x+dx, y+dy, z+dz)$ be consecutive points on a surface of the congruence derived from (A) we have

$$\frac{\partial F_r}{\partial x}dx + \frac{\partial F_r}{\partial y}dy + \frac{\partial F_r}{\partial z}dz = 0, \qquad (r = 1, 2).$$

Hence the normal at (x, y, z) to the surface is given by the directions $\left(\frac{\partial F_r}{\partial x}, \frac{\partial F_r}{\partial y}, \frac{\partial F_r}{\partial z}\right)$ where r may have *either* of its values. This means that the point is a double point, unless the surfaces F_1 and F_2 happen to touch there.

An alternative argument leading to the same conclusion is as follows: If S denote the surface of the congruence generated by the

* § 287. Forsyth. *loc. cit.*

curves $F_r(x, y, z\ a, \phi(a))=0$, $r=1, 2$, then in general S touches the surface F_r for any definite values of a and r at the points common to $S=0$, $F_r=0$, $E_r=0$ where $E_r=0$ is the envelope of the system $F_r=0$. I have proved this result elsewhere*, but the result is true only when the common point is not a double point on $E_r=0$, or on $S_r=0$, and when some other exceptional features are absent. We have therefore the following possibilities :

(i) Either, S touches both F_1 and F_2 at the points common to the four surfaces in (A), i.e. F_1 and F_2 touch each other at these points.

(ii) Otherwise, the points are in general double points on one or other of the surfaces $S=0$, $E_1=0$, $E_2=0$.

That (i) does not give the general case is easily seen, for, eliminating x, y, z from

$$F_r(x, y, z, a, b)=0, \qquad (r=1, 2)$$

$$\frac{\partial F_1}{\partial x} : \frac{\partial F_1}{\partial y} : \frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial x} : \frac{\partial F_2}{\partial y} : \frac{\partial F_2}{\partial z}$$

we obtain a relation $\psi(a, b)=0$. In case this happens to be a solution of $\lambda\left(a, b, \frac{db}{da}\right)=0$, we obtain a *single* surface of the congruence which satisfies (i), and which touches the focal surface.

To conclude, the ∞^1 of the surfaces of the congruence defined in § 1 will not touch the focal surface, as a rule, but isolated exceptions may exist.

§ 3. One or two special types of congruences will now be examined.

Consider the congruence $f_1(x, y, z, a)=0$, $f_2(x, y, z, b)=0$ where each of the equations involves only one of the two constants a and b. We may put these equations in the form $u(x, y, z)=a$, $v(x, y, z)=b$. There are two possibilities :

Case (1). The envelopes of the two systems $u=a$ and $v=b$ may be different. If E denotes the envelope of, say, the system $u=a$, then E touches in general any surface of the congruence, of the form $\phi(u, v)=0$, all along their curve of intersection.† But the surfaces $v=b$, which should also be regarded as surfaces of the congruence, do not touch E, except perhaps, for special values of b. This case is

* *Tohoku Math. Journ.* Vol. 39.

† Vide *Tohoku Math. Journ.* loc. cit.

therefore exactly analogous to that of the general congruence discussed earlier. The envelopes of the two systems $u=a$ and $v=b$ together constitute the focal surface.

Case (2). The two systems $u=a$ and $v=b$ may have the same envelope. In this case, u and v must involve the same irrational or transcendental expressions. It is only in rare instances that we can find a function $\phi(u, v)$ which is free from these expressions (Ex. 4. furnishes such an instance). Therefore the common envelope of the two systems constitutes the focal surface, and *touches, in general, all surfaces of the congruence, without exception.*

[The more general forms $F_r(x, y, z, a, b)=0$, $r=1, 2$ can be reduced to the forms considered in this section. It can be proved* that case (2) usually presents itself. But the curve $F_1=0$, $F_2=0$ is only a part of the curve of intersection of the corresponding $u=a$, $v=b$. The surfaces of the congruence $F_1=0$, $F_2=0$ will therefore be different from the (non-degenerate) surfaces of the congruence $u=a$, $v=b$, although the two congruences possess the same focal surface.]

§ 4. From the discussion so far, one useful result can be deduced :

If any singly infinite system (except, in general, one particular system) of surfaces of the congruence possesses a unique (i.e. non-degenerate or single) envelope, this envelope constitutes the focal surface or part of it.

One of the general properties of the focal surface is that it touches each curve of the congruence at each of the focal points. This property however breaks down in one special case, viz. Case (2) of § 3. When $u=a$ and $v=b$ possess a common envelope $E=0$, then at a point of intersection of the three surfaces $E=0$, $u=a$, $v=b$, the surfaces touch one another. *The focal surface in this case should therefore be regarded as the locus of the double points of the curves of the congruence, and not as the envelope of these curves.*

§ 5. EXAMPLES†.

EXAMPLE 1.
$$F_1 \equiv -y+ax+a^2+b=0$$
$$F_2 \equiv -z+bx+ab=0$$

Eliminating x, y, z between the equations (A), the general solution of the resulting differential equation is $b=ac+c^2$, where c is an arbitrary

* See my paper in *Proc. Indian. Acad. Sci.* Vol. I, No. 10. Example 5.

† These examples have been selected for the purpose from the works of Forsyth and M. J. M. Hill where they are used for other purposes, in connection with singular solutions of differential equations.

constant. Using this with $F_1=0$ and $F_2=0$, we find that all the developable surfaces of the congruence (except one, which corresponds to the singular solution of the resulting differential equation) are plane surfaces given by $z=c^2x+cy-c^3$. Fixing the value of c, the focal points on the corresponding surface are given by

$$x=-2a-c\,;\; y=c^2-a^2\,;\; z=-c(a+c)^2.$$

It is easy to verify that at this point, the equation to the tangent plane to the focal surface

$$(27z-9xy-2x^3)^2=4\ (x^2+3y)^3$$

is not free from a, i.e. the plane $z=c^2x+cy-c^3$ does not touch the focal surface at the point for any value of a.

EXAMPLE 2.

$$F_1\equiv -y+ax-9b^2+18a^2b-4a^4=0.$$
$$F_2\equiv -z+bx+12ab^2-4a^3b=0.$$

Eliminating x, y, z as before from the equations (A), we obtain, *either* $b=a^2$, or a differential equation whose general solution is $b=ac-\frac{3}{2}c^2$ where c is an arbitrary constant. We thence find that all, except two*, of the developable surfaces of the congruence are planes, viz., $z-cy+\frac{3}{2}c^2x=\dfrac{81}{4}c^5$. The co-ordinates of the focal points on one of these planes (for a definite value of c) are

$$x=16a^3-54a^2c+72ac^2-27c^3$$

$$y=12a^4-36a^3c+36a^2c^2-\frac{81}{4}c^4.$$

$$z=12a^4c-60a^3c^2+117a^2c^3-108ac^4+\frac{81}{2}c^5.$$

Taking a and c for curvilinear co-ordinates, these equations determine the focal surface. It may be actually verified that the equation of the tangent plane is different from the plane

$$z-cy+\tfrac{3}{2}c^2x=\frac{81}{4}c^5.$$

EXAMPLE 3. The congruence

$$u\equiv y+2(z-x-y)^{\frac{1}{2}}=a\,;\; v\equiv z-2y=b$$

has $z=x+y$ as the focal surface. It is the envelope of the system $u=a$, and touches all surfaces of the congruence except those given by $v=b$.

* These two correspond to the isolated factor $b=a^2$, and to the singular solution of the differential equation obtained after elimination. In either of these cases, the locus of the focal points on the corresponding surface is found to be a double line.

EXAMPLE 4. The congruence

$$u \equiv x + y(z - x - y)^{\frac{1}{2}} = a; \quad v \equiv y + x(z - x - y)^{\frac{1}{2}} = b$$

has $z = x + y$ as the focal surface. This is the common envelope of $u = a$ and $v = b$, and touches all surfaces of the congruence except those given by $\phi(u, v) \equiv u^2 - v^2 = \text{constant}$.

EXAMPLE 5. The congruence

$$u \equiv x + y(z - x - y)^{\frac{1}{2}} = a; \quad v \equiv x^2 + y^2(z - x - y)^{\frac{1}{2}} = b$$

has $z = x + y$ for the focal surface which touches all the surfaces of the congruence without exception.

EXAMPLE 6.

$$u \equiv \frac{x - y}{1 + xy} = a; \quad v \equiv \frac{y(1 - z^2)^{\frac{1}{2}} - z}{yz + (1 - z^2)^{\frac{1}{2}}} = b.$$

The two planes given by $z^2 = 1$ form the focal surface. They touch all surfaces of the congruence except those given by $u = a$.

ON LINEAR QUOTIENT-SEQUENCES

BY

HANSRAJ GUPTA, *Hoshiarpur*.

1. Introduction.

(*a*) The quotient obtained by dividing a positive integral number x by a positive integral number y has been denoted as usual by $[x/y]$; the remainder so obtained has been denoted by $R_y(x)$.

(*b*) Two sequences (u_r) and (u'_r), $r = 1, 2, 3, \ldots\ldots, n$; are said to possess a t-correspondence when

$$u_t = u'_1, \; u_{t+1} = u'_2, \; u_{t+2} = u'_3, \; \ldots\ldots \; u_n = u'_{n-t+1}, \; u_1 = u'_{n-t+2}, \; \ldots\ldots\ldots, \; u_{t-1} = u'_n.$$

The existence of such a correspondence is expressed by writing

$$(u_t) \equiv (u'_1). \qquad [1]$$

If the terms of two such sequences be arranged in order round two concentric circles, then by turning round the circle with (u_r) until u_t comes into correspondence with u'_1, the two sequences can be made to agree term for term. The value of t can be easily found by this device.

2. The Problem.

In the first part of the paper some properties of a Linear Quotient-Sequence given by $\left[\frac{kr+l}{m}\right]$, $r=1, 2, 3, \ldots\ldots$, are stated; k, m being positive integers, and l a positive or negative integer, $|l| \leqslant k$ in the latter case.

These are then made use of in solving the converse problem viz.

"Given a set of n positive integral numbers $a_1, a_2, a_3, \ldots, a_n$, such that $a_r - a_{r-1} = d$ or $d+1$, to express, if possible, the given set as a linear quotient-sequence modulo m, where $m \leqslant n$".

The case $m > n$ has also been considered.

The possibility of such a representation was noticed in my paper* entitled "The Perpetual Calendar Formula".

The quotient-sequence given by $\left[\frac{kr+l}{m}\right]$, $r = 1, 2, 3, \ldots\ldots$, has been termed linear mod. m, because of its rise from a linear expression $kr+l$ on division by m. The quotient-sequence $\left[\frac{hr^2+kr+l}{m}\right]$ may be termed quadratic.

3. Some Properties of Linear Quotient-Sequences.

(i) The linear quotient-sequence (a_r) given by

$$a_r = \left[\frac{kr+l}{m}\right] \quad , r = 1, 2, 3, \ldots\ldots$$

is "Periodic" with a periodicity m (or a sub-multiple of m, when $\{k, m\} > 1$.)

Thus we have $a_{m+r} = k + a_r$. [2]

(ii) Connected with the sequence (a_r) is a 'remainder-sequence' (b_r) where $b_r = \mathrm{R}_m\,(kr + l)$.

This has the same periodicity as (a_r) and we have

$$b_{r+m} = b_r. \qquad [3]$$

(iii) The difference between consecutive members of (a_r) is $d+1$, or d, according as there is or there is not a decrease in (b_r); i.e., $a_r - a_{r-1} = d$ or $d + 1$, according as $b_r - b_{r-1}$ is positive or negative, $r \geqslant 2$.

* *Journal of the Indian Mathematical Society:* April, 1926.

We take $a_1 - a_0 = d$ or $d + 1$, according as $b_1 - b_0$, i.e. $b_{m+1} - b_m$ is positive or negative.

(iv) Connected with (a_r) is also a rank-sequence (c_r) defined by the relations:

$$c_r = a_r - a_{r-1} - d, \qquad r \geqslant 2,$$

and
$$c_1 = a_1 - a_0 - d = a_{m+1} - a_m - d;$$

so that $c_r = 0$ or 1, according as $a_r - a_{r-1} = d$ or $d + 1$. [4]

Moreover $c_r = 0$ or 1, according as $b_r - b_{r-1}$ is positive or negative. [5]

The sequence (c_r) has the same periodicity as (a_r); and we have $c_{m+r} = c_r$.

(v) The rank-sequences connected with (a_r) and (a'_r)

where $$a'_r = \left[\frac{kr}{m}\right], \; r = 1, 2, 3, \ldots\ldots$$

possess a t-correspondence.

In fact $(c'_t) \equiv (c_1)$. [6]

4. The Converse Problem.

Let $a_1, a_2, a_3, \ldots a_n$ be the given set of n numbers, such that $a_r - a_{r-1} = d$ or $d+1$. If possible let $a_r = \left[\frac{kr+l}{m}\right]$, $m \leqslant n$. First we find k.

(i) If $m < n$, then we must have

$$a_{r+m} - a_r = k, \qquad r = 1, 2, 3, \ldots\ldots, n - m.$$

If this condition, which is necessary but not sufficient for the representation, is not satisfied, then the solution is impossible for the modulus m. If the condition is satisfied the problem reduces to that of expressing the first m terms of the set by a linear quotient-sequence mod. m, with $k = a_{m+1} - a_1$.

This is but a special case of (ii) below.

(ii) If $m = n$, we suppose $a_r = \left[\frac{kr+l}{m}\right]$.

Then $$k = a_{n+1} - a_1 = (a_n + d) - a_1, \quad \text{or } (a_n + d + 1) - a_1. \qquad [7]$$
so that k can only have one or the other of the two values.

Moreover $(a_1 + 1)\, n > k + l \geqslant a_1\, n$.

Second to find l in the main case.

Determine the rank-sequence (c_r) connected with the given set. (Take $c_1 = 0$ or 1, according as $k = a_n + d - a_1$, or $a_n + d + 1 - a_1$) Determine also the rank-sequence (c'_r) connected with $\left(\left[\frac{kr}{m}\right]\right)$, $1 \leqslant r \leqslant n$.

If $(c_t') \equiv (c_1)$, then $k + l = a_1 n + R_n\ (kt)$; so that $l = a_1 n - k + R_n(kt)$.

If $(c_t') \neq (c_1)$, then the solution is not possible for mod. n. Hence the necessary and sufficient condition that the converse problem may be solvable, is the existence of a t-correspondence in the sequences (c_r') and (c_r), $1 \leqslant r \leqslant n$.

(iii) When $m > n$, we have

$a_n + (m-n)\, d \leqslant a_m \leqslant a_n + (m-n)\ (d+1)$,

and $a_n + (m-n+1)\ d \leqslant a_{m+1} \leqslant a_n + (m-n+1)\ (d+1)$.

This gives us a range of $(m-n+2)$ values for k. Each of these can be considered in turn. The terms between a_n and a_m are partially under our control in this case.

5. Example.

Express, if possible, the set of numbers :

2, 5, 7, 10, 12, 15, 18, 20, 23, 25, 28, 31,

as a linear quotient-sequence ; (i) mod. 5, (ii) mod. 11, (iii) mod. 13.

(i) Here $a_6 - a_1 = a_7 - a_2 = \cdot\ \cdot\ \cdot\ \cdot = a_{12} - a_7 = 13$.

Therefore $k = 13 = a_5 + d + 1 - a_1$, where $d = 2$. Hence $c_1 = 1$.

The sequence (c_r) is 1, 1, 0, 1, 0.

The sequence (c'_r) is 0, 1, 0, 1, 1.

$(c'_t) \equiv (c_1)$ for $t = 4$.

Hence $l = 2 \times 5 - 13 + R_5\ (13 \times 4) = -1$.

The required quotient-sequence is $\left[\frac{13r-1}{5}\right]$, $1 \leqslant r \leqslant 12$.

(ii) Here $k = a_{12} - a_1 = 29$, and $a_{12} = a_{11} + d + 1$.

Therefore $c_1 = 1$.

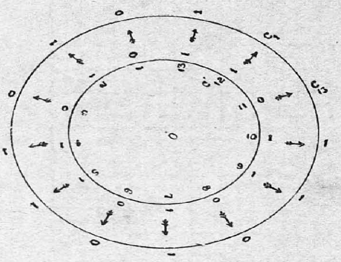
The sequence (c_r) is 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1.

The sequence (c'_r) is 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1.

Since $(c_t) \neq (c'_1)$ for any t; therefore the given set cannot be expressed as a quotient-sequence modulo 11.

(iii) Since $a_{14} = 35$, 36, or 37; therefore $k = a_{14} - a_1 = 33$, 34, or 35. The sequences (c_r) and (c'_r) are:

$r =$	1	2	3	4	5	6	7	8	9	10	11	12	13	
(c_r)	c_1	1	0	1	0	1	1	0	1	0	1	1	c_{13}	$k = 33, 34, 35.$
	0	1	0	1	0	1	0	1	0	1	0	1	1	$k = 33, (c'_t) \neq (c_1).$
(c'_r)	0	1	0	1	1	0	1	0	1	1	0	1	1	$k = 34.$
	0	1	1	0	1	1	0	1	1	0	1	1	1	$k = 35, (c'_t) \neq (c_1).$



When $k = 34$, $(c'_t) \equiv (c_1)$ provided $c_{13} = 0$, and $c_1 = 1$; (this will be the case if we take $a_{14} = 36$, and $a_{13} = 33$.)

We thus get $t = 12$. Hence $l = 2 \times 13 - 34 + R_{13}\ (34 \times 12) = -3$.

The required quotient-sequence is $\left[\frac{34r - 3}{13}\right]$, $1 \leqslant r \leqslant 13$.

6. General remarks.

(i) If $c_r = 1$, when $r = j_1, j_2, j_2, \cdots, j_s$; $j_{h-1} < j_h$.

Then the sequence (j_h), $h = 1, 2, 3, \cdots, s$, is such that

$$j_h - j_{h-1} = v, \text{ or } v + 1.$$

This condition is necessary but not sufficient for the representations discussed in this paper.

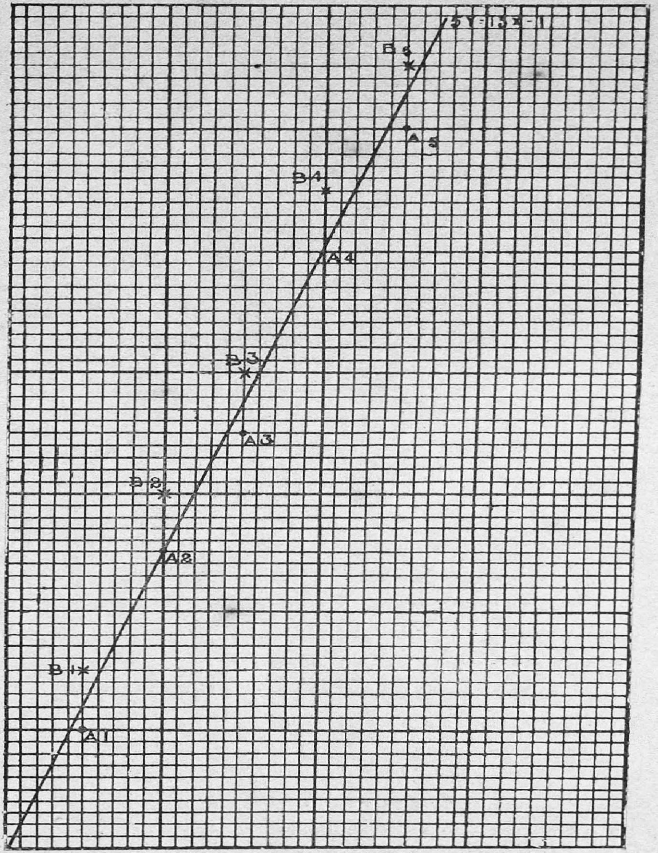
(ii) If $(c_{i+1}) \equiv (c'_1)$, then (8) can be written $l = a_i n - ki$; [8a] where $i = n - t + 1$.

(iii) If we plot the points;

$$\left.\begin{array}{ll}(A) & x = nr, \ y = na_r; \\ (B) & x = nr, \ y = n(a_r + 1);\end{array}\right\} \quad r = 1, 2, 3, \cdots, n.$$

then in order that the set of numbers $a_1, a_2, a_3, \cdots, a_n$, may be expressible as a linear quotient-sequence mod. n, it is necessary and sufficient that it should be possible to draw a line $ny = kx + l$,

k, l, being integers; such that all the points (B) lie on one side of it, and the points (A) lie upon it or on the other side of it.



FOCI IN COMPLEX GEOMETRY

BY

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This note is the continuation of a previous paper on the subject in the *Journal of the Indian Mathematical Society*, Vol. XVII (1928) pp. 223-232.

1. The foci of a curve of class $(n-1)$ touching all the mutual joins of n points possess properties analogous to the foci of a conic inscribed in a triangle.

(i) The general equation of the curve is

$$\frac{k_1}{\pi_1} + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \frac{k_n}{\pi_n} = 0 \qquad \dots \quad (1)$$

where $k_1 \cdots k_n$ are real numbers and $\pi_1, \pi_2 \cdots \pi_n$ are the tangential equations (in rectangular cartesian co-ordinates) of the n points $A_1 \cdots A_n$.

The $(n-1)$ foci are given by

$$\sum_{r=1}^{r=n} \frac{k_r}{z-\alpha_r} = 0. \qquad \dots \quad (2)$$

where $z, \alpha_1 \cdots \alpha_n$ are respectively the complex numbers representing the vectors to the focus and to the n given points.

Let $\beta_1 \cdots \beta_{n-1}$ be the $(n-1)$ foci $B_1 \cdots B_{n-1}$; then

$$\sum k_1 (z-\alpha_2) \cdots (z-\alpha_n) \equiv (k + \cdots k_n)(z-\beta_1) \cdots (z-\beta_{n-1}). \quad \dots \quad (3)$$

Writing $z = \alpha_1$ etc. we get n equalities of the type

$$k_1(\alpha_1-\alpha_2) \cdots (\alpha_1-\alpha_n) = (\sum k) \cdot (\alpha_1-\beta_1) \cdots (\alpha_1-\beta_{n-1}) \qquad \dots \quad (4)$$

Equating the moduli and arguments we get

(i) $k_1 \cdot A_1 A_2 \cdots A_1 A_n = A_1 B_1 \cdot A_1 B_2 \cdots A_1 B_{n-1} (\sum k)$ ... (5)

(ii) The sum of the inclinati on of $A_1 A_2, \cdot A_1 A_n$ to a fixed line = the sum of the inclinations of $A_1 B_2 \cdots A_1 B_{n-1}$. ... (6)

$$\text{Also} \quad \frac{k_1(\alpha_1-\alpha_2) \cdots (\alpha_1-\alpha_n)}{k_2(\alpha_2-\alpha_1) \cdots (\alpha_2-\alpha_n)} = \frac{(\alpha_1-\beta_1) \cdots (\alpha_1-\beta_{n-1})}{(\alpha_2-\beta_1) \cdots (\alpha_2-\beta_{n-1})}. \qquad \dots \quad (7)$$

$$\text{i.e.} \left(-\frac{k_1}{k_2}\right) \cdot \left(\frac{\alpha_1-\alpha_3}{\alpha_2-\alpha_3}\right) \cdots \left(\frac{\alpha_1-\alpha_n}{\alpha_2-\alpha_n}\right) = \left(\frac{\alpha_1-\beta_1}{\alpha_2-\beta_1}\right) \cdots \left(\frac{\alpha_1-\beta_{n-1}}{\alpha_2-\beta_{n-1}}\right) \dots \quad (8)$$

$$\therefore \quad (2m+1)\pi + A_1\hat{A}_3A_2 + \cdots A_1\hat{A}_nA_2 = A_1\hat{B}_1A_2 + \cdots A_1\hat{B}_{n-1}A_2. \quad \dots \quad (9)$$

The properties (5) and (9) are analogous to those of a pair of isogonal conjugates with respect to a triangle.

2. When one or more of the foci lie at infinity i.e. when the curve touches the line at infinity, the properties still hold good.

When $(n-2)$ foci lie at infinity, the curve touches the line at infinity at $(n-2)$ points. Such a curve is the $(n-1)$ fold parabola of Clifford.* Let M be the focus of the unique $(n-1)$ fold parabola inscribed in the n-gon. Then we have

$$k_1 \cdot A_1 A_2 \cdot \cdot \cdot \cdot A_1 A_n = A_1 M \cdot \lambda \qquad \dots \quad (10)$$

$$(2m+1)\pi + A_1 \hat{A}_3 A_2 \; . \; . \; . \; A_1 \hat{A}_n A_2 = A_1 \hat{M} A_2. \qquad \dots \quad (11)$$

Here λ is a constant and is not equal to $k_1 + k_2 + \cdot \cdot \cdot \cdot k_n$.

The above are similar to those obtained for the quadrangle in my former paper.

When the five points $A_1 A_2 A_3 A_4 A_5$ consist of the vertices of a triangle and a pair of isogonal conjugates, whose join passes through the circumcentre, the point M for the 4-fold parabola inscribed in the 5-gon is the circumcentre.

3. The position of M in a n-gon may be determined in another manner. For equation (2), $(n-2)$ roots are infinite. Expanding the equation, we have

$$\sum_{r=1}^{r=n} k_r \left(\frac{1}{z} + \frac{\alpha_r}{z^2} + \cdot \cdot \cdot \frac{\alpha_r^{\,n}}{z^{n+1}} + \cdot \cdot \right) = 0$$

$$\therefore \quad \sum k_r = 0 \qquad \sum k_r \alpha_r = 0 \cdot \cdot \cdot \qquad \sum k_r \alpha_r^{\,n-3} = 0. \qquad \dots{}^{**}(12)$$

When the $(n-2)$ conditions are satisfied, there remains the single root μ, which is the required point.

Write $\lambda_r = \dfrac{k_r}{\mu - \alpha_r}$

$$\therefore \quad \sum \lambda_r = 0 \; ; \; \cdot \cdot \cdot \cdot \quad \sum \lambda_r \alpha_r^{\,n-2} = 0. \qquad \dots \quad (13)$$

$$\therefore \quad \frac{\lambda_1}{\zeta^{1/2}(\alpha_2 \cdot \cdot \alpha_r)} = \text{similar fractions.} \qquad \dots \; \dagger(14)$$

Where $\zeta^{1/2}(\alpha_2 \cdot \cdot \alpha_r)$ = product of all the $\frac{1}{2}(n-1)(n-2)$ differences of $\alpha_2 \cdot \cdot \alpha_r$.

Also $\mu = (-\sum k_1 \alpha_2 \alpha_3 \cdot \cdot \cdot \alpha_n) / (\sum k_1 \alpha_1^{\,n-2})$. (Wolstenholme) ... (15)

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* CLIFFORD: *Collected Mathematical Papers, p. 49.*

** Vide: Wolstenholme's *Problem.* No. 180, 3rd Edition.

† Scott and Matthew's *Determinants.* p. 149.

4. *The configuration of n points which have a final M point.*

The condition (12) may be written out in full (write $x_r + iy_r = \alpha_r$) as follows :—

$$\sum k_r = 0$$
$$\sum k_r x_r = \sum k_r y_r = 0$$
$$\sum k_r(x_r^2 - y_r^2) = \sum k_r 2x_r y_r = 0$$
$$\sum k_r(x_r^3 - 3x_r y_r^2) = \sum k_r(3x_r^2 y_r - y_r^3) = 0$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots \quad \ldots$$
$$\sum k_r \cdot {}_r u_{n-3} = \sum k_r \cdot {}_r v_{n-3} = 0 \cdot \cdot \cdot \qquad \ldots \quad (16)$$

where ${}_r u_{n-3}$, ${}_r v_{n-3}$ are conjugate homogeneous polynomials in $x_r\ y_r$ of degree $(n-3)$.

We have $(n-1)$ ratios $k_1 : k_2 : \cdots k_n$ to be eliminated from $2n-5$ equations and we will get $n-4$ relations between the n base points. If $(x+iy)^n = u_n + iv_n$, we deduce that the base points are points of the intersections of curves of the following type :

$$A_{n-3}(u_{n-3} + c_{n-3}\, v_{n-3}) + \cdot \cdot \cdot + A_1(x + c_1\, y) + A_0 = 0 \qquad \ldots \quad (17)$$

where the A's and c's are constants.

(*a*) When $n=4$, there is no special configuration. When M is at infinity, the four points are the intersection of two rectangular hyperbolas of the type

$$A_2(u_2 + c_2 v_2) + A_1(x + c_1 y) + A_0 = 0.$$

(*b*) When $n=5$, and M is at a finite distance, the five points lie on the rectangular hyperbola.

$$A_2(u_2 + c_2 v_2) + A_1(x + c_1 y) + A_0 = 0.$$

When $n=5$ and M is at infinity, we find, on eliminating $k_1 \cdot \cdot k_5$ that the base points are the points of intersection of a rectangular hyperbola and a cubic whose equation is of the type

$$u_3 + cv_3 + Ax + By + C = 0.$$

i.e. whose asymptotes are parallel to the sides of an equilateral triangle.

RECURRENCE FORMULAE FOR BERNOULLI'S NUMBERS

BY

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Introduction

1. Recurrence Formulæ for the Bernoullian Numbers have been constructed by Bernoulli, Stern and others in various ways. The detailed account will be found in the book, *Traite Elementaire des Nombres de Bernoulli* by Niels Nielsen, Paris (1923).

In this paper further formulæ for the Bernoullian numbers, complete (i.e. connecting $B_1, B_2, B_3, \cdots B_p$) as well as incomplete (i.e., connecting the B's in some such form as $B_n, B_{n-1}, \cdots B_p$, p being $n/2$ or $(n\pm1)/2$ according as n is even or odd), are investigated with the aid of Symmetric Functions and Determinants, most of the formulæ being simpler than the existing ones.

Determinants of the type $\{n^n\}$ and $\{n^n l\}$, as defined in Art. 3 and 4, are evaluated. Incidentally some interesting relations in Binomial Coefficients are shown to exist.

I wish to acknowledge my great indebtedness to Prof. A. R. Richardson for his suggestions and kind supervision of my work.

2. It is well known* that

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + B_1\frac{z^2}{2\,!} - B_2\frac{z^4}{4\,!} + B_3\frac{z^6}{6\,!} - + \cdots, \tag{1}$$

where B's are the Bernoullian numbers,

$$B_1 = \frac{1}{6},\quad B_2 = \frac{1}{30},\quad B_3 = \frac{1}{42},\quad B_4 = \frac{1}{30},\ \text{etc.}$$

Let $f(z) = (1-az)(1-bz)(1-cz)\cdots(1-wz)$.

$$\begin{aligned} &= a_0 - a_1 z + a_2 z^2 - \cdots + (-1)^r a_r z^r + \ldots, \qquad (2)\\ &= 1 - \{1\}z + \{1^2\}z^2 - \cdots + (-1)^r\{1^r\}z^r + \ldots; \end{aligned}$$

then,

$$\begin{aligned} [f(z)]^{-1} &= h_0 + h_1 z + h_2 z^3 + \cdots + h_r z^r + \ldots, \qquad (3)\\ &= 1 + (1)z + (2)z^3 + \cdots + (r)z^r + \ldots, \end{aligned}$$

where a_r and h_r are the elementary and complete ... homogeneous symmetric functions in $a, b, c, \ldots$ of degree r.

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* Whittaker and Watson, *Modern Analysis* (1915) p. 127.

Take the B's for the a's, then comparing (1) and (2) we shall have $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{B_1}{2\,!}$, $a_3 = 0$, $a_4 = -\frac{B_2}{4\,!}$ $a_5 = 0$, $a_{2n} = (-1)^{n \pm 1} \frac{B_n}{(2n)\,!}$, and $a_{2r+1} = 0$.

Since $\frac{e^z - 1}{z} = 1 + \frac{z}{2\,!} + \frac{z^2}{3\,!} + \cdots$ it is clear from (3) that $h_r = 1/(r+1)\,!$

3. Formula for $\{n^n\}$.*

Consider the Determinant

$$\{n^n\} = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{2n-1} \\ a_{n-1} & a_n & \cdot & \cdot & \cdot & \cdot & \cdot & a_{2n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

By Kostka's **Theorem the determinant with elements a's is equal to the determinant with elements h's, if the diagonal suffixes of the one determinant and those of the other are conjugate partitions.

Hence

$$\overline{\{n^n\}} = \{n^n\} = \begin{vmatrix} h_n & h_{n+1} & \cdots\cdots & h_{2n-1} \\ h_{n-1} & h_n & \cdots\cdots & h_{2n-2} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ h_1 & h_2 & \cdots\cdots & h_n \end{vmatrix}$$

the bar —, denoting conjugate partitions.

$$\text{i.e.} \quad \{n^n\} = \begin{vmatrix} \frac{1}{(n+1)\,!} & \frac{1}{(n+2)\,!} & \cdots & \frac{1}{(2n)\,!} \\ \frac{1}{n\,!} & \frac{1}{(n+1)\,!} & \cdots & \frac{1}{(2n-1)\,!} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{2\,!} & \frac{1}{3\,!} & \cdots & \frac{1}{(n+1)\,!} \end{vmatrix}$$

To find the value multiply the first row of the determinant by $(2n)\,!$ second by $(2n-1)\,!$ and so on, last row by $(n+1)\,!$. Subtract the elements of each row from those of the above, continuing the process of subtraction and dividing finally by $(n+1)\,!\,(n+2)\,!\ldots\ldots(2n)\,!$ we get the result.

$$\{n^n\} = \frac{(n-1)\,!\,(n-2)\,!\,\ldots\ldots 2\,!}{(n+1)\,!\,(n+2)\,!\,\ldots\ldots(2n)\,!}.$$

* [The notation is evidently based on the suffixes of the diagonal terms —ED.]

** Muir. *Contribution to the History of Determinants* (1930). P. 194.

This formula was suggested by Prof. Richardson.
Thus also,

$$\{n^m\} = \frac{(m-1)!\,(m-2)!\,\ldots\,2!}{(n+1)!\,(n+2)!\,\ldots\,(n+m)!}.$$

4. Evaluation of $\{n^n l\}$.

Consider another type of determinant

$$\{n^n l\} = \begin{vmatrix} \frac{1}{(n+1)!} & \frac{1}{(n+2)!} & \cdots & \frac{1}{(2n)!} & \frac{1}{(2n+1)!} \\ \frac{1}{n!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(2n-1)!} & \frac{1}{(2n)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(n+1)!} & \frac{1}{(n+2)!} \\ 0 & 0 & \cdots & \frac{1}{l!} & \frac{1}{(l+1)!} \end{vmatrix}$$

$l<n$; when $l \geqslant n$, there will be no zeros in the last row.

Expanding in terms of the elements of the last row there will be, in general, minors of the form,

$$D = \begin{vmatrix} \frac{1}{p!} \cdots \frac{1}{(p+s-1)!}, & \frac{1}{(p+s+1)!} \cdots \frac{1}{(p+n)!} \\ \frac{1}{(p-1)!} \cdots \frac{1}{(p+s-2)!}, & \frac{1}{(p+s)!} \cdots \frac{1}{(p+n-1)!} \\ \cdots & \cdots \\ \frac{1}{(p-n+1)!} \cdots \frac{1}{(p-n+s)!}, & \frac{1}{(p-n+s+2)!} \cdots \frac{1}{(p+1)!} \end{vmatrix}$$

Multiply the first row by $(p+n)!$, second by $(p+n-1)!$ and so on, the last row by $(p+1)!$

Perform the process of subtraction of each row from the one above, $(n-s)$ times, then taking the common factors out, the result is

$$D = \frac{(p+1)(p+2)\ldots(p+s)\,.\,n!(n-1)!\ldots(s+1)!}{(p+1)!\,(p+2)!\ldots(p+n)!\,.\,(n-s)!}$$

multiplied by a determinantal factor which can be easily found to be

$$(s-1)!\,(s-2)!\,\ldots\,1.$$

Hence

$$D = \frac{n!\ldots(s+1)!\,(s-1)!\ldots3!\,2!\,.\,(p+1)(p+2)\ldots(p+s)}{(n-s)!\,(p+1)!\ldots(p+n)!}$$

Since D is known $\{n^n l\}$, or $\{n^m l\}$ can be evaluated.

Example.

$$\{4^3 1\} = \begin{vmatrix} \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{8!} \\ \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} \\ \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} \\ 0 & 0 & 1 & \frac{1}{2!} \end{vmatrix}$$

$$= \frac{1}{2!} \{4^3\} - 1. \begin{vmatrix} \frac{1}{5!} & \frac{1}{6!} & \frac{1}{8!} \\ \frac{1}{4!} & \frac{1}{5!} & \frac{1}{7!} \\ \frac{1}{3!} & \frac{1}{4!} & \frac{1}{6!} \end{vmatrix}$$

$$= \frac{1}{2!} \cdot \frac{2!}{5!\,6!\,7!} - \frac{3!\cdot 6\times 7}{6!\,7!\,8!} = \frac{84}{6!\,7!\,8!}.$$

5. Relations in Binomial Coefficients.

Some interesting relations of type A exist in the binomial coefficients, which will frequently occur in constructing the recurrence formulae.

A. $$\binom{2n}{n+1} - \binom{k}{1}\binom{2n-1}{n+1} + \binom{k}{2}\binom{2n-2}{n+1} \cdots\cdots$$
$$+ (-1)^{k-1}\binom{k}{k-1}\binom{2n-k+1}{n+1} + (-1)^k\binom{2n-k}{n+1} = \binom{2n-k}{n-1}$$

where k is $2s$ or $2s+1$, s taking the values, 0, 1, 2, . . .

For, the left hand side is the coefficient of x^{n+1} in
$$(1+x)^{2n} - \binom{k}{1}(1+x)^{2n-1} \cdots + (-1)^k(1+x)^{2n-k} = (1+x)^{2n-k}\Big[(1+x)-1\Big]^k$$
so that it is the coefficient of x^{n+1-k} in $(1+x)^{2n-k}$. Hence the result.

I am thankful to Dr. Aicken (Edinburgh) for suggesting the method of proof.

More generally
$$\binom{2n+1}{n+m} - \binom{k}{1}\binom{2n+l-1}{n+m} + \binom{k}{2}\binom{2n+l-2}{n+m} \cdots + (-1)^k\binom{2n+l-k}{n+m}$$
$$= \binom{2n+l-k}{n+m-k}$$

where k is $2s$ or $2s+1$ and l and m, have the values, 0, 1, 2, . . .

6. Bernoulli's Formula.

We shall obtain here the following formula* due to Bernoulli

$$\sum_{s=1}^{s \leqslant (m-1)/2} (-1)^{s-1} \binom{m}{2s} B_s = \frac{m-2}{2}$$

Consider the Determinant

$$\{n\ 1^n\} = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots\cdots & a_{2n-1} & a_{2n} \\ 1 & a_1 & a_2 & \cdots\cdots & a_{n-1} & a_n \\ 0 & 1 & a_1 & \cdots\cdots & a_{n-2} & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots\cdots & 1 & a_1 \end{vmatrix}$$

Expand, *when n is even*

$$a_n\{\overline{1^n}\} + a_{n+2}\{\overline{1^{n-2}}\} + \cdots\cdots + a_{2n} = \{\overline{n}\,\overline{1^n}\}.$$

the bar —, denoting conjugate partitions;

$$= (-1)^{n/2-1}\frac{B_{n/2}}{n\,!}(n) + (-1)^{n/2}\frac{B_{n/2+1}}{(n+2)\,!}(n-2) + \cdots\cdots$$

$$+ (-1)^{n-1}\frac{B_n}{(2n)\,!} = \{n+1,\ 1^{n-1}\}.$$

$$= (-1)^{n/2-1}\frac{B_{n/2}}{n\,!\,(n+1)\,!} + (-1)^{n/2}\frac{B_{n/2+1}}{(n+2)\,!\,(n-1)\,!} + \cdots\cdots$$

$$+ (-1)^{n-1}\frac{B_n}{(2n)\,!} = \{n+1,\ 1^{n-1}\} \qquad \ldots(4)$$

Now

$$\{n+1,\ 1^{n-1}\} = \begin{vmatrix} \frac{1}{(n+2)\,!} & \frac{1}{(n+3)\,!} & \cdots\cdots & \frac{1}{(2n)\,!} & \frac{1}{(2n+1)\,!} \\ 1 & \frac{1}{2\,!} & \cdots\cdots & \frac{1}{(n-1)\,!} & \frac{1}{n\,!} \\ 0 & 1 & \cdots\cdots & \frac{1}{(n-2)\,!} & \frac{1}{(n-1)\,!} \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & 1 & \frac{1}{2\,!} \end{vmatrix}$$

$$= \left[-\frac{1}{(n+3)\,!}\{1^{n-2}\} - \frac{1}{(n+5)\,!}\{1^{n-4}\} - \cdots\cdots + \frac{1}{(2n)\,!}\{1\} - \frac{1}{(2n+1)\,!}\right]$$

$$= -\left[(-1)^{n/2-2}\frac{B_{n/2-1}}{(n+3)\,!\,(n-2)\,!} + \cdots + \frac{B_1}{(2n-1)\,!\ 2\,!}\right] + \frac{1}{2\cdot(2n)\,!} - \frac{1}{(2n+1)\,!}$$

* Niels Nielsen's Book p. 151.

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Hence (4) becomes

$$\sum_{s=1}^{n}(-1)^{s-1}\binom{2n+1}{2s}B_s = \frac{2n-1}{2}.$$

which is Bernoulli's formula when $2n+1=m$.

When n *is odd,* the expansion will give,

$$\left[(-1)^{(n-1)/2}\frac{B_{(n+1)/2}}{(n-1)!\,n!}+(-1)^{(n+1)/2}\frac{B_{(n+3)/2}}{(n+3)!\,(n-2)!}+\cdots+(-1)^{n-1}\frac{B_n}{(2n)!}\right]$$

$$= - \{n+1,\ 1^{n-1}\}$$

Proceeding as above, we immediately obtain the formula. Hence the formula of Bernoulli is established.

7. Recurrence Formulae

We shall now construct the formulae for the Bernoullian numbers. These formulae in general, with the exception of a few, hold true for values of $n \geqslant 2$.

Consider the Determinant,

$$\{n^{n+1}\} = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{2n} \\ a_{n-1} & a_n & \cdot & \cdot & \cdot & a_{2n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_{n+1} \\ 1 & a_1 & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

Expanding in terms of the elements of the last column, when n is even,

$$a_{2n}\overline{\{(n-1)^n\}} + a_{2n-2}\overline{\{n^2(n-1)^{n-2}\}} + \cdots\cdots + a_n\overline{\{n^n\}} = \overline{\{n^{n+1}\}}.$$

i.e. $$(-1)^{n+1}\frac{B_n}{(2n)!}\{n^{n-1}\} + (-1)^n\frac{B_{n-1}}{(2n-2)!}\{n^{n-1}2\} + \ldots\ \ldots\ \ldots$$

$$+(-1)^{n/2+2}\frac{B_{n/2+1}}{(n+2)!}\{n^{n-1}\,n-2\} + (-1)^{n/2+1}\frac{B_{n/2}}{n!}\{n^n\} = \{(n+1)^n\}.$$

Using the results of Art. 3 and 4, we have,

$$\frac{(-1)^{n+1}B_n}{(2n)!}\left[\frac{(n-2)!\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 1!}{(n+1)!\ \cdot\ \cdot\ \cdot\ (2n-1)!}\right]+$$

$$\frac{(-1)^n B_{n-1}}{(2n-2)!}\left[\begin{array}{ll} \frac{1}{3!} & \frac{(n-2)!\ \cdot\ \cdot\ \cdot\ 1}{(n+1)!\ \cdot\ \cdot\ \cdot\ (2n-1)!} \\ -\frac{1}{2!} & \frac{(n-1)!\ (n-3)!\ \cdot\ \cdot\ 2!\ (n+2)(n+3)\ \cdot\ \cdot\ (2n-1)}{(n+2)!\ (n+3)!\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ (2n)!} \\ + & \frac{(n-1)!\ (n-2)!\ (n-4)!\cdot\ \cdot\ 1(n+2)(n+3)\cdot\cdot(2n-2)}{2!\ (n+2)!\ (n+3)!\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ (2n)!} \\ + & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right]$$

$$+(-1)^{n/2+2}\frac{B_{n/2+1}}{(n+2)!}\left[\frac{1}{(n-1)!}\frac{(n-2)!\cdots\cdots 2!}{(n+1)!\cdots\cdots(2n-1)!}-\frac{1}{(n-2)!}\times\frac{(n-1)!\,(n-3)!\cdots\cdot 1\cdot(n+2)(n+3)\cdots\cdot(2n-1)}{(n+2)!\,(n+3)!\cdots\cdots\cdots(2n-1)!\,(2n)!}\right.$$

$$+\;\cdots\cdots\cdots\cdots$$

$$\left.+1\cdot\frac{(n-1)!\,(n-2)!\cdots\cdots 2!\,(n+2)}{(n-2)!\,(n+2)!\,(n+3)!\cdots\cdots(2n)!}\right]$$

$$+(-1)^{n/2+1}\frac{B_{n/2}}{n!}\left[\frac{(n-1)!\cdots\cdots 1}{(n+1)!\cdots\cdots(2n)!}\right]$$

$$=\frac{(n-1)!\cdots\cdots\cdots 1}{(n+2)!\cdots\cdots(2n+1)!}.$$

Multiply throughout by $\frac{(n+2)!\,(n+3)!\cdots\cdots(2n)!}{(n-1)!\,(n-2)!\cdots\cdots 2\cdot 1}$.

then,

$$(-1)^{n+1}\frac{B_n}{(2n)!}\binom{2n}{n+1}$$

$$+(-1)^n\frac{B_{n-1}}{(2n-2)!}\left[\frac{1}{3!}\left[\binom{2n}{n+1}-\binom{3}{1}\binom{2n-1}{n+1}+\binom{3}{2}\binom{2n-2}{n+1}\right]\right.$$

$$+\;\cdots\cdots\cdots\cdots$$

$$+(-1)^{n/2+2}\frac{B_{n/2+1}}{(n+2)!}\frac{1}{(n-1)!}\left[\binom{2n}{n+1}-\binom{n-1}{1}\binom{2n-1}{n+1}+\cdots+\binom{n-1}{n-2}\binom{n+2}{n+1}\right]$$

$$+(-1)^{n/2+1}\cdot\frac{B_{n/2}}{n!}\cdot\frac{1}{(n+1)!}=\frac{1}{(2n+1)!}.$$

Since from A. of Art. 5.

$$\binom{2n}{n+1}-\binom{2s+1}{1}\binom{2n-1}{n+1}+\cdots\cdots+\binom{2s+1}{2s}\binom{2n-2s}{n+1}$$

$$=\binom{2n-2s-1}{n-1}+\binom{2n-2s-1}{n+1}.$$

the whole series is easily summed up and the formula obtained is for n *even*:

$$\text{I.}\qquad \sum_{t=0}^{n/2}(-1)^{n-t+1}\left[\binom{2n-2t-1}{n-1}+\binom{2n-2t-1}{n+1}\right]\binom{2n+1}{2t+1}B_{n-t}=1.$$

When n is odd, the corresponding formula, is,

$$\sum_{t=0}^{(n-1)/2}(-1)^{(n-t)+1}\cdot\binom{2n+1}{2t+1}\left[\binom{2n-2t-1}{n-1}+\binom{2n-2t-1}{n+1}\right]B_{n-t}=-1.$$

II. Replace the last column of the Determinant by the last but one column, then by the expansion of the determinant, when n is even,

$$(-1)^n \frac{B_{n-1}}{(2n-2)!}\{n^{n-1}1\}+(-1)^{n-1}\frac{B_{n-2}}{(2n-4)!}\{n^{n-1}3\}+\cdots$$

$$(-1)^{n/2+1}\frac{B_{n/2}}{n!}\{n^{n-1}\,n-1\}=0.$$

Since

$$\binom{2n}{n+1}-\binom{2s}{1}\binom{2n-1}{n+1}+\cdots\binom{2s}{2s-1}\binom{2n-2s+1}{n+1}=\binom{2n-2s}{n-1}-\binom{2n-2s}{n+1}.$$

the result derived is

$$\sum_{t=1}^{n/2}(-1)^{n-t+1}\binom{2n}{2t}\left[\binom{2n-2t}{n-1}-\binom{2n-2t}{n+1}\right]B_{n-t}=0.$$

for n even, and

$$\sum_{t=1}^{(n+1)/2}(-1)^{n-t+1}\binom{2n}{2t}\left[\binom{2n-2t}{n-1}-\binom{2n-2t}{n+1}\right]B_{n-t}=0$$

when n is odd.

III. Let the first column take also the place of the last column, then the expansion will give,

$$(-1)^{n/2+1}\frac{B_{n/2}}{n!}\left\{n^{n-1}\right\}+(-1)^{n/2}\frac{B_{n/2-1}}{(n-2)!}\left\{n^{n-1}\,2\right\}+\cdots.$$

$$+\frac{B_1}{2!}\left\{n^{n-1}\,n-2\right\}-\tfrac{1}{2}\left\{n^{n-1}\,n-1\right\}+\left\{n^{n}\right\}=0.$$

which, when summed up with the help of A of Art. 5, will yield

$$\sum_{t=0}^{n/2-1}(-1)^{n/2-t+1}\binom{n+1}{2t+1}\left[\binom{2n-2t-1}{n-1}+\binom{2n-2t-1}{n+1}\right]B_{n/2-t}$$

$$-\frac{n}{2}(n+1)+1=0.$$

when n is even, and

$$\sum_{t=1}^{(n-1)/2}(-1)^{(n-2t+3)/2}\cdot\left[\binom{2n-2t}{n-1}-\binom{2n-2t}{n+1}\right]\binom{n+1}{2t}B_{(n-2t+1)/2}$$

when n is odd $$-\frac{n(n+1)}{2}+1=0.$$

IV. Expand the Determinant in terms of the elements of the first column, then, when n is even,

$$(-1)^{n/2+1}\frac{B_{n/2}}{n!}\left\{n^{n}\right\}+(-1)^{n/2}\frac{B_{n/2-1}}{(n-2)!}\left\{n^{n}\,2\right\}+\cdots.$$

$$+\frac{B_1}{2!}\left\{n^{n}\,n-2\right\}-\tfrac{1}{2}\left\{n^{n}\,n-1\right\}+\left\{n^{n+1}\right\}=\left\{(n+1)^{n}\right\}.$$

Summing the series with the help of

$$\binom{2n+1}{n+1}-\binom{k}{1}\binom{2n}{n+1}+\cdots\cdots+(-1)^k\binom{2n+1-k}{n+1}=\binom{2n+1-k}{n}$$

the formula obtained is

$$\sum_{t=0}^{n/2-1}(-1)^{n/2-t+1}\binom{n+1}{2t+1}\binom{2n-2t+1}{n+1}B_{n/2-t}=\frac{n}{2}(n+3)$$

for n even and

$$\sum_{t=1}^{(n-1)/2}(-1)^{(n-2t+3)/2}\binom{2n-2t+1}{n}\binom{n}{2t-1}B_{(n-2t+1)/2}=\frac{(n-1)(n+2)}{2}$$

for n odd.

V. Replace the first column by the last column, then by expansion and summation we shall have, for n even,

$$\sum_{t=0}^{n/2}(-1)^{n-t+1}\binom{2n+1}{2t+1}\binom{2n-2t+1}{n+1}B_{n-t}=0$$

and

$$\sum_{t=0}^{(n-1)/2}(-1)^{n-t+1}\binom{2n+1}{2t+1}\binom{2n-2t+1}{n+1}B_{n-t}=0.$$

for n odd.

VI. Let the second column take the place of the first column; then expanding in terms of its elements and summing,

$$\sum_{t=1}^{n/2}(-1)^{n/2-t+2}\cdot\binom{n+2}{2t}\left[\binom{2n-2t+1}{n}-\binom{2n-2t+1}{n+1}\right]B_{n/2-t+1}-\frac{n+2}{2}=0,$$

for n even, and

$$\sum_{t=0}^{n-1/2}(-1)^{(n-2t+3)/2}\cdot\binom{n+2}{2t+1}\binom{2n-2t+1}{n+1}B_{(n-2t+1)/2}=\frac{n}{2}+1.$$

for n odd.

VII. Replace the first column by any other column, say the fifth, then by expansion, etc. we arrive at

$$\sum_{t=0}^{n/2}(-1)^{n/2-t+3}\binom{n+5}{2t+1}\binom{2n-2t+1}{n+1}B_{n/2-t+2}=0$$

for n even and >2, and

$$\sum_{t=1}^{(n+1)/2}(-1)^{(n-2t+7)/2}\cdot\binom{n+5}{2t}\left[\binom{2n-2t+1}{n}-\binom{2n-2t+1}{n+1}\right]B_{(n-2t+5)/2}=0$$

for n odd and >3.

VIII. Let the last column be replaced by any other column say the fourth, then expanding in terms of its elements and summing up as before, we shall get the formula, for $n>3$.

$$\sum_{t=1}^{n/2}(-1)^{n/2-t+3}\cdot\binom{n+4}{2t}\left[\binom{2n-2t}{n-1}-\binom{2n-2t}{n+1}\right]B_{n/2-t+2}=0$$

for n *even*, and

$$\sum_{t=0}^{(n-1)/2}(-1)^{(n-2t+5)/2}\cdot\binom{n+4}{2t+1}\left[\binom{2n-2t-1}{n-1}+\binom{2n-2t-1}{n+1}\right]B_{(n-2t+3)/2}=0.$$

when n is odd

IX. Consider the determinant of the $(n+2)$th order, $\{n^{n+2}\}$.

Expanding the determinant in terms of the elements of the last column, then when n is even,

$$(-1)^{n+1}\frac{B_n}{(2n)!}\left\{(n+1)^{n-1}\quad 1\right\}+(-1)^n\frac{B_{n-1}}{(2n-2)!}\left\{(n+1)^{n-1}\quad 3\right\}+\cdots$$
$$+(-1)^{n/2+1}\frac{B_{n/2}}{n!}\left\{(n+1)^n\right\}=\left\{(n+2)^n\right\}.$$

Forming the series, simplifying and summing as in Art. 7 formula I, we obtain the rusult

$$\sum_{t=1}^{n/2+1}(-1)^{n-t+2}\left[\binom{2n-2t+1}{n-1}-\binom{2n-2t+1}{n+2}\right]\binom{2n+2}{2t}B_{n-t+1}=1.$$

for n even, and

$$\sum_{t=1}^{n+1/2}(-1)^{n-t+?}\left[\binom{2n-2t+1}{n-1}-\binom{2n-2t+1}{n+2}\right]\binom{2n+2}{2t}B_{n-t+1}=-1.$$

for n odd.

X. Take the last column of the determinant the same as the last but one, then after expansion and summation as in IX, we shall get,

$$\sum_{t=0}^{n/2}(-1)^{n-t+1}\left[\binom{2n-2t}{n-1}+\binom{2n-2t}{n+2}\right]\binom{2n+1}{2t+1}B_{n-t}=0. \qquad n \text{ even},$$

$$\sum_{t=0}^{(n+1)/2}(-1)^{n-t+1}\left[\binom{2n-2t}{n-1}+\binom{2n-2t}{n+2}\right]\binom{2n+1}{2t+1}B_{n-t}=0. \qquad n \text{ odd}.$$

XI. Replace the last column by the first column, expand in terms of its elements, then proceeding as above, we shall obtain the relation :

$$\sum_{t=0}^{n/2-1}(-1)^{n/2-t+1}\left[\binom{2n-2t}{n-1}+\binom{2n-2t}{n+2}\right]\binom{n+1}{2t+1}B_{n/2-t}-\frac{n}{4}\left[(n+1)^2-4\right]=0.$$

for n even and

$$\sum_{t=1}^{(n-1)/2}(-1)^{(n-2t+3)/2}\left[\binom{2n-2t+1}{n-1}-\binom{2n-2t+1}{n+2}\right]\binom{n+1}{2t}B_{(n-2t+1)/2} = \frac{n}{4}\left[(n+1)^2-4\right].$$

for n odd.

XII. Expand the determinant in terms of the elements of the first column, then after summation of the series with the help of the relation

$$\binom{2n+2}{n+2}-\binom{k}{1}\binom{2n+1}{n+2}+\cdots\cdots+(-1)^{k-1}\binom{k}{k-1}\binom{2n+3-k}{n+2} = \binom{2n+2-k}{n}-(-1)^k\binom{2n+2-k}{n+2},$$

k being $2s+1$ or $2s$, the formula obtained is

$$\sum_{t=0}^{n/2-1}(-1)^{n/2-t+1}\left[\binom{2n-2t+1}{n}+\binom{2n-2t+1}{n+2}\right]\binom{n+1}{2t+1}B_{n/2-t} = \frac{1}{4}(n+1)(n^2+3n)$$

for n *even*, and

$$\sum_{t=1}^{(n-1)/2}(-1)^{(n-2t+3)/2}\left[\binom{2n-2t+2}{n}-\binom{2n-2t+2}{n+2}\right]\binom{n+1}{2t}B_{(n-2t+1)/2} = \frac{(n+1)(n^2+3n-4)}{4}.$$

for n *odd*.

Other formulae may be similarly constructed.

GENERAL FORMULAE FOR CONSTRUCTING AND SOLVING CERTAIN SIMULTANEOUS EQUATIONS *

BY

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1. Let

$$a_1^n + a_2^n + \cdot \cdot \cdot a_k^n = b_1^n + b_2^n \cdot \cdot \cdot + b_k^n \qquad \ldots \quad \text{(A)}$$

be true for $n = 1, 2, 3 \ldots (2p+1)$, and t any number. Then

$$\sum (t - b_r)^n + \sum (t + a_r)^n = \sum (t + b_r)^n + \sum (t - a_r)^n \qquad \ldots \quad \text{(B)}$$

for $n = 1, 2, \ldots 2p+1, 2p+2$ the summation for r being from 1 to k.

2. If (A) holds for $n = 1, 3, 5 \ldots (2p+1)$, then (B) holds for $n = 1, 2, 3, 4, 5 \ldots (2p+2)$. Thus from

$$\dagger\ (3, 9, 21, 21)^n = (1, 13, 17, 23)^n \qquad (n = 1, 3, 5)$$

we have on taking $t = 24$

$$(1, 7, 11, 23, 27, 33, 45, 45)^n = (3, 3, 15, 21, 25, 37, 41, 47)^n$$
$$n = 1, 2, 3, 4, 5, 6.$$

3. If the relation (A) holds for $n = 1, 2, \ldots 2p$, then

$$\sum (t - b_r)^n + \sum (t + b_r)^n = \sum (t - a_r)^n + (t + a_r)^n \qquad \ldots \quad \text{(C)}$$

for $n = 1, 2 \ldots 2p+1$, the summation for r being from 1 to k, as usual.

4. If (A) holds for $n = 2, 4, 6 \ldots 2p$, then (C) holds for $n = 1, 2, 3, 4, 5 \ldots 2p+1$.

5. Obviously Equation (A) will continue to hold if we replace $a_1, a_2 \ldots b_1, b_2 \ldots$ by $ca_1, ca_2 \ldots cb_1, cb_2 \ldots$ or by $a_1 \pm c, a_2 \pm c, \ldots b_1 \pm c, b_2 \pm c, \ldots$ the upper or lower sign being taken in each case. By taking c equal to a_1 or $a_2 \ldots$, we may reduce the number of terms to $k-1$ on the left side.

6. If (A) holds for $n = 1, 2, 3 \ldots 2p$, then we have at most a constant C defined by

$$C = a_1 + b_k = a_2 + b_{k-1} = \ldots a_k + b_1$$

provided a_r and b_r are arranged in ascending order. Taking $c = \frac{1}{2}C$ in § 5, we have $\sum(\frac{1}{2}C - a_r)^n = (\frac{1}{2}C - b_r)^n$ and the negative terms in $\frac{1}{2}C - a_r$ will occur as positive terms in $\frac{1}{2}C - b_r$. Thus

$$(2, 10, 20, 32, 54, 56, 76, 78)^n = (4, 6, 26, 28, 50, 62, 72, 80)^n$$
$$n = 1, 2, 3, 4, 5, 6.$$

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* [Translated from the original in German.—*Ed.*]

† Read $(3^n + 9^n + 21^n + 21^n = 1^n + 13^n + 17^n + 25^n)$.

Here $C = 2 + 80 = 10 + 72 = \cdot \cdot = 82$ and we take $\frac{1}{2}C - a_r = 41 - a_r$. Hence

$$(39, 31, 21, 9, -13, -15, -35, -37)^n = (37, 35, 15, 13, -9, -21, -31, -39)^n \quad n = 1, 2, 3, 4, 5, 6$$

whence

$$(39, 31, 21, 29)^n = (37, 35, 15, 13)^n \qquad (n = 1, 3, 5.)$$

7. If (A) holds for $n = 1, 2, \ldots 2p + 1$, and we have

$$a_1 + a_k = a_2 + a_{k-1} = \ldots = b_1 + b_k = b_2 + b_{k-1} = \ldots C \qquad \text{(say)},$$

we may proceed as before replacing a_r and b_r by $\frac{1}{2}C - a_r$ and $\frac{1}{2}C - b_r$.

8.* We shall now obtain a general solution of (A) for $k = 8$, $n = 1, 2, 3, 4, 5, 6$.

The identity

$$2(i^2 + 3k^2)^n = (2ik + 3k^2 - i^2)^n + (2ik + i^2 - 3k^2)^n + (4ik)^n, \qquad (n = 2, 4),$$

gives solutions of $0^n + A^n + A^n = B^n + C^n + D^n \qquad (n = 2, 4)$

whence by § 4,

$$(t - A, t - A, t, t, t + A, t + A)^n = (t - D, t - C, t - B, t + B, t + C, t + D)^n$$

for $n = 1, 2, 3, 4, 5$ which gives relations of the required form on using the result in § 2. Thus take $i = 2$, $k = 1$ giving

$$(0, 7, 7)^n = (3, 5, 8)^n \quad n = 2, 4.$$

With $t = 9$, we have

$$(2, 2, 9, 9, 16, 16)^n = (1, 4, 6, 12, 14, 17)^n \qquad (n = 1, 2, 3, 4, 5).$$

Subtract twice each term from 25, and we have

$$(21, 21, 7, 7, -7, -7)^n = (23, 17, 13, 1, -3, -9)^n \qquad (n = 1, 2, 3, 4, 5).$$

Therefore $(3, 9, 21, 21)^n = (1, 13, 17, 23)^n \qquad (n = 1, 3, 5)$.

Now using § 2 with $t = 24$, we have

$$(1, 7, 11, 23, 27, 33, 45, 45)^n = (3, 3, 15, 21, 25, 37, 41, 47)^n$$

for $n = 1, 2, 3, 4, 5, 6$ as required.

9. The Equation (A) may be solved in integers for $n = 1, 2$ when all the terms on one side are given in advance. Thus given all the a's, we have if $\sigma = 2(a_1 + \cdots a_k)/k$,

$$(a_1, \ldots a_k)^n = (\sigma - a_k, \ldots \sigma - a_1)^n \qquad (n = 1, 2).$$

By thus choosing suitably the elements of one side we can solve equations like

$$(a_1^v, a_2^v, \ldots a_k^v)^n = (b_1, \ldots b_r)^n \qquad (n = 1, 2).$$

* [The author remarks in a letter that the results in this para are not new.—*Ed.*]

Thus $(9^3)^n = 729^n + 0^n + 0^n$

$$= (-243,\ 486,\ 486)^n \quad \text{since } \sigma = \tfrac{2}{3} \times 729 = 486.$$

10. If $(a_1\ a_2\ .\ .\ .\ a_k)^n = (b_1\ b_2\ .\ .\ .\ b_k)^n = (c_1\ .\ .\ .\ c_k)^n \qquad n = 2, 4\ .\ .\ 2p$

then

$$\sum (t - a_r)^n + \sum (t + a_r)^n = \sum (t - b_r)^n + \sum (t + b_r)^n$$

$$= \sum (t - c_r)^n + \sum (t + c_r)^n = \ .\ .\ .\ .$$

$$(n = 1, 2, 3, \ .\ .\ .\ 2p + 1)$$

by the repeated use of the theorem of § 3.

Thus from

$146575^2 + 799500^2 = 178425^2 + 793000^2 = 223199^2 + 780312^2 = 312625^2 + 750300^2 = 0^2 + 812825^2$ and $t = 812826$, we may obtain a number of relations $(a_1, a_2, a_3, a_4)^n = (b_1, b_2, b_3, b_4)^n = (c_1, c_2, c_3, c_4)^n = \ .\ .\ . \qquad n = 1, 2, 3$

11. *Problems and conjectures*

(i) The equation $a_1{}^n + a_2{}^n = b_1{}^n + b_2{}^n$ is conjectured to be solvable for every integral n; but so far, solutions are known for $n < 5$. What are the solutions for $n \geqslant 5$.

(ii) The equation $(a_1, a_2, a_3)^n = (b_1, b_2, b_3)^n$ appears to be solvable for $n = n_1, n_2$ and hence for both degrees simultaneously.

Thus: $(12,\ 16,\ 17)^n = (13,\ 14,\ 18)^n \qquad n = 1, 2$

$(7,\ 25,\ 35)^n = (11,\ 19,\ 37)^n \qquad n = 1, 3$

$(1,\ 1777,\ 2218)^n = (623,\ 974,\ 2399)^n \qquad n = 1, 4$

$(123,\ 389,\ 442)^n = (208,\ 272,\ 474)^n \qquad n = 1, 5$

In all these examples one of the values of n is unity. In the following example the values are 2 and 4

$$(1, 9, 10)^n = (5, 6, 11)^n \qquad n = 2, 4.$$

(iii) The equation $a^n = b_1{}^n + b_2{}^n + \ .\ .\ .\ b_k{}^n$ appears to be insoluble in rational numbers for $k < n$.

This conjecture, due to Euler, generalises Fermat's famous theorem which corresponds to the value $k = 2$. The equation

$$a^n = b_1{}^n + b_2{}^n + \ .\ .\ .\ b_n{}^n$$

is now solved for $n \leqslant 5$, *general* solutions being available for $n = 2, 3$ and special solutions for $n = 4$ and 5 namely

$$353^4 = 30^4 + 120^4 + 272^4 + 315^4.$$

$$107^5 = 7^5 + 43^5 + 57^5 + 80^5 + 100^5.$$

Herr Norrie who discovered the result for $n=4$ has informed me that he discovered another example of this type several years ago, but has not published it yet. There remains the question of finding solutions for $n > 5$.

(iv) The equation

$$a_1^n + \ldots a_k^n = b_1^n + \ldots b_k^n \qquad n = 1, 2, 3, \ldots p$$

has non-trivial solutions only when $p \geqslant k$. We have seen (§ 5) that the least term a_1 may be reduced to zero. Thus

$$a_1^n + a_2^n = b_1^n + b_2^n + b_3^n \qquad (n=1, 2)$$

is solvable and has the solution

$$(7, 8)^n = (2, 3, 10)^n \qquad (n=1, 2).$$

(v) In the equality

$$(4, 8, 10, 15, 19, 19)^n = (5, 6, 11, 16, 17, 20)^n \qquad (n=1, 2, 4, 6)$$

the values of n do not form an arithmetic series. This raises the question of the general solution of such equations.

NOTES AND DISCUSSIONS

On the expression of a Biquadrate as the sum of three biquadrates

In a note with the same title in the last issue of the *Student*,* Mr. Brojomohan Padhy has noticed an error in Werebrusow's proof of Euler's conjecture of the impossibility of

$$x^4 + y^4 + z^4 = u^4.$$

He observes that in Werebrusow's equation

$$(3+4t^2)^4 = (-1+4t^2)^4 + (2+4t)^4 + 16[2(2t-1)\ (2t^2 + 2t+1)]^2$$

the last term should be $16[2(2t-1)(2t^2+t+1)]^2$ and that, "in this form of the equation, it does not seem easy to prove that the last term cannot be made a biquadrate". This can, however, readily be established in an elementary manner, as follows.

Since

$$2t^2+t+1 = (2t-1)\ (t+1)+2,$$

$(2t^2+t+1)$ and $(2t-1)$ are prime to each other. In order that $16[2(2t-1)\ (2t^2+t+1)]^2$ may be a biquadrate, it is therefore necessary

* *Mathematics Student*, Vol. III, No. 3, pp. 100-101.

that each of the expressions $(2t-1)$ and $2\ (2t^2+t+1)$ should be a perfect square. Now for all positive integral values of t

$$(2t)^2 < 2(2t^2+t+1) < (2t+1)^2\ ;$$

and so $2(2t^2+t+1)$, being intermediate in value between the squares of two consecutive integers, cannot itself be a perfect square. From this result, as indicated by Mr. Padhy, the proof of Euler's conjecture follows.

New Delhi : R. KOTHANDARAMAN.

On the nature of the contact of the conics $S=0$ and $S+\lambda T=0$.

When two curves touch each other, the contact may be conveniently called *internal* or *external* according as the two centres of curvature at that point lie on the same side or on the opposite sides of the common tangent.

The object of this note is to prove:

THEOREM I: Let $S=0$, be the equation to any conic and $T=0$, the equation to the tangent at any point on the conic; then the contact of the conic with the conic $S-\lambda T=0$ is internal or external according as $\lambda < 2$ or > 2; provided T is formed according to the usual rule and its form is preserved.

While infinity is well recognised as a limiting value for λ, the fact that λ has a finite limiting value which is independent of the coefficients of S, appears to have remained unnoticed.

The above can be proved otherwise, but with the application of calculus may be proved the following general Theorem (II) of which the above is only a particular case.

THEOREM II: If

$$S_1 \equiv F(x,\ y) = 0 \tag{1}$$

$$F(x_1,\ y_1) = 0 \tag{2}$$

$$T_1 \equiv x\ \partial F/\partial x_1 + y\ \partial F/\partial y_1 - x_1\ \partial F/\partial x_1 - y_1\ \partial F/\partial y_1 \tag{3}$$

$$S_2 \equiv S_1 - \lambda T_1 = 0 \tag{4}$$

then the two curves (1) and (4) touch each other at $(x_1,\ y_1)$ and the contact is internal or external according as $\lambda < 1$ or $\lambda > 1$.

(If $S \equiv S_1$, then it may be noted that in Theorem I, $T = T_1/2$.)

For the sake of convenience, let us write (4) as

$$F(X, Y) - \lambda\{XF_{x1} + YF_{y1} - x_1F_{x1} - y_1F_{y1}\} = 0. \tag{5}$$

Differentiating twice (1) and (5) with respect to x and X respectively, we get

$$F_x - F_y \frac{dy}{dx} = 0. \tag{6}$$

$$F_y \frac{d^2y}{dx^2} = -\left\{ F_{xx} + 2F_{xy}\frac{dy}{dx} + F_{yy}\left(\frac{dy}{dx}\right)^2 \right\} \tag{7}$$

$$F_X + F_Y \frac{dY}{dX} - \lambda F_{x1} - \lambda F_{y1} \frac{dY}{dX} = 0. \tag{8}$$

$$(F_Y - \lambda F_{\ 1}) \frac{d^2Y}{dX^2} = -\left\{ F_{XX} + 2F_{XY}\frac{dY}{dX} + F_{YY}\left(\frac{dY}{dX}\right)^2 \right\} \tag{9}$$

When $X = x = x_1$ and $Y = y = y_1$, from (6) and (8), we get

$$\left(\frac{dy}{dx}\right)_{x=x1} = \left(\frac{dY}{dX}\right)_{X=x1}. \tag{10}$$

Further (5) vanishes when $X = x_1$, $Y = y_1$. Hence the two curves touch each other at (x_1, y_1).

Again, from (10), when $X = x = x_1$, $Y = y = y_1$, the expressions on the right hand sides of (7) and (9) become equal. Therefore, if

$$A = \left(\frac{d^2y}{dx^2}\right)_{x=x1} \quad \text{and} \quad B = \left(\frac{d^2Y}{dX^2}\right)_{X=x1}, \tag{11}$$

then from (7) and (9), by substituting $X = x = x_1$, $Y = y = y_1$, we get that

$$Fy_1.\ A = (Fy_1 - \lambda Fy_1).\ B$$

that is

$$\frac{A}{B} = 1 - \lambda. \tag{12}$$

Since the sign of the second differential coefficient decides whether the gradient of the tangent increases or not in the neighbourhood of the point, the two curves will have their convexities towards the same side or not, according as $\frac{A}{B}$ is positive or negative. Hence from (12), the Theorem II follows.

Annamalainagar : S. S. PILLAI.

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BOULIGAND, GIRAUD, and DELENS: *Le problème de la dérivée oblique en théorie du potential*, Actualities Scientifiques ₂ No. 219, Herman and Co., Paris, 1935.

WEIL: *Arithmétique et Géométrie sur les variétés algébriques*; No. 206, 1935.

PAUL DUBREIL: *Quelques propriétés des variétés algébriques se rattachant aux théories de l'algèbre moderne* No. 210, 1935.

NEUMANN: *Integraloperators*, No. 229, 1935.

D. R. KAPREKAR: *Ten Cuts in Calculations*; pp. 48, Arya Waibhav Press, Nasik, Price Annas Six.

ANNOUNCEMENTS AND NEWS

The following have been admitted as members of the Society:

Durga Prasad Banerjee Esq., M.A., Professor, A. M. College, Mymensingh.

Miss S. Pankajam, M.A., 97, Big Street, Triplicane, Madras.

K. Nagabhushanam Esq., M.A, Lecturer, Andhra University, Waltair.

Rev. C. Racine, D.Sc. (Paris), Professor, St Joseph's College, Trichinopoly.

Sukumar Ranjan Das Esq., M.A., Ph.D. (Cal.), Professor, Commercial College, Delhi.

Harivansh Lall Rishi Esq., M.A., Professor, D. A. V. College, Hoshiarpur.

Hari Shankar Esq., M.A., Lecturer, Anglo-Arabic College, Delhi.

Mr. H. G. Gharpurey, I.C.S., resigned his office as President of the Indian Mathematical Society in November last owing to ill health. Dr. R. P. Paranjpye, Vice-Chancellor of the Lucknow University, has been elected President of the Society by the managing Committee.

Prof. S. B. Belekar, Secretary of the Society, resigned his office and Dr. Ram Behari, Professor, St. Stephen's College, has been elected Secretary in his place.

In place of Prof. A. C. Banerjee who resigned his membership of the Society, Dr. S. Mukhopadhyaya, Retired Professor, Calcutta University, has been elected a member of the Committee.

The ninth Conference of the Indian Mathematical Society was inaugurated by Sir Girija Shankar Bajpai, Member for Education, Health and Lands, at Delhi on the 18th December 1935. Dr. Vaidyanathaswami delivered the Presidential address on the morning of the 19th, while the evening was devoted to a business meeting of the Society. Besides 29 papers which were contributed to the conference, there were public lectures on "Science and Fatalism" by Dr. G. S. Mahajani, "The Place of Mathematics in Present-day Culture" by Dr. Vaidyanathaswami and on "Cosmography" by Prof. A. C. Banerjee. There was, as usual, one evening devoted to a discussion on the teaching of Mathematics in Schools and Colleges. The delegates who numbered about 80 and the elite of the City were the guests at several garden parties organised by Rai Bahadur Ram Kishore, Vice-Chancellor, Delhi University, Khan Bahadur Dr. Sir Mohammad Abdur Rahman, and the members of the Reception Committee. The detailed proceedings of the conference will be published later.

The Senate of the Calcutta University has appointed Dr. Friedrich Levi as head of the Department of Pure Mathematics for 5 years.

Dr. S. Chowla, Reader and Head of the Mathematics Department, Andhra University, has accepted an appointment in the Government College, Lahore.

Dr. S. Chandrasekhar, Fellow of Trinity College, Cambridge, has been appointed lecturer in Cosmic Physics at Harvard University.

Mr. W. V. D. Hodge, Lecturer in Mathematics and Fellow of Pembroke College, has been appointed to succeed Prof. H. F. Baker as Lowndean Professor of Astronomy and Geometry in the University of Cambridge.

The gold medal of the Royal Astronomical Society has been awarded, to Prof. H. Kimura, Director of the International latitude observatory at Mizusawa, Japan. His principal discovery relates to the existence of a small variation that is constant with respect to the longitude of the station but varies with the latitude. This is the first time that the medal is awarded to one outside Europe or North America.

We very much regret to announce the death of V. B. Naik, the Librarian of our Society and Professor of Mathematics at the Fergusson College, Poona.

QUESTIONS FOR SOLUTION

1724. (VICTOR LEVIN)—Shew that if n is an integer $\geqslant 1$, the determinant of the n-th order

$$\Delta_n = \| \delta_{\lambda\mu} \|,$$

where $\delta_{\lambda\mu}$ is the number of combinations of $(n+1)$ things $2\lambda - \mu$ at a line $(1 \leqslant \lambda, \mu \leqslant n)$,

has the value $$2^{\frac{1}{2}n^2 + \frac{1}{2}n},$$

provided that all combinatory symbols ${}_pC_q$ with $q < 0$ and $q > p$ are replaced by zeros.

1725. (C. T. RAJAGOPAL)—If $\sum u_n$ is a divergent series of positive terms, discuss the behaviour of the series $\sum u_n/(1 + n^s u_n)$.

1726. (V. THEBAULT)—a, b, c, B are four positive integers in arithmetic progression, such that $aacc = (bb)^2$, the base of the system of numeration being B. Find possible values for B.

1727. (A. A. KRISHNASWAMI AYYANGAR)—If A, B, C, A′, B′, C′ be any six coplanar points such that the circles ABC′, BCA′, CAB′, A′B′C′ have a common point D, prove that the circles A′B′C, B′C′A, C′A′B, and ABC also have a common point, say D′.

If the six given points lie on a conic Γ passing through D, prove that the centres of the above *eight* circles lie on a rectangular hyperbola, whose asymptotes are the axes of Γ. Show that D′ also must lie on Γ.

1728. (A. NARASINGA RAO)—Show that for any tangential pencil of quadrics, the tangent planes to the three quadrics through a point and the four faces of the common self polar tetrahedron are osculating planes of the same twisted cubic.

1729. (A. A. KRISHNASWAMI AYYANGAR)—"Given that the length of the shadow of a vertical pole 12 feet long, 4 hours after sunrise, is 9 feet at a place where the equinoctial shadow is 5 feet, determine the longitude of the Sun." (Bhaskhara)

(This problem is of historical interest, having been solved by Bhaskhara by successive approximations in his *Ganitadhyaya*. It is, however, capable of an exact solution.)

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